NORTH-HOLLAND

Block Analogies of Comparison Matrices

Miroslav Fiedler* and Vlastimil Pták
Institute of Computer Science
Czech Academy of Sciences
Pod vodárenskou věží 2
182 07 Praha 7, The Czech Republic

Submitted by Richard A. Brualdi

ABSTRACT

A Hadamard-type comparison matrix $H(A)$ is defined for block matrices A. With this notion, an extension to block matrices is given for Ostrowski's inequality $|\det A| \geqslant \det H(A)$. A related notion of equimodularity is introduced, and a block analogy of the result of Camion and Hoffman is proved: all matrices equimodular with a given A are nonsingular if and only if, after a permutation, $H(A)$ is a (nonsingular) M-matrix. © Elsevier Science Inc., 1997

1. INTRODUCTION

The notion of an M-matrix has a long and interesting history. It seems that the starting point was a simple but important observation by Hadamard: A matrix with dominant diagonal is nonsingular. More precisely, if (a_{ik}) is a complex n-by-n matrix such that $|a_{ii}| - \sum_{k \neq i}|a_{ik}| > 0$ for each i, then $\det A \neq 0$. The following step in the development was a result of Minkowski on diagonal dominance which permits a stronger assertion about the determinant to be made: If for $M = (m_{ik})$, $m_{ii} \geqslant 0$ for all i and $m_{ik} \leqslant 0$ for $i \neq k$, and M is diagonally dominant, then $\det M > 0$. A. Ostrowski observed that a weaker form of diagonal dominance is sufficient for this result to remain valid. In fact, it suffices to assume diagonal dominance for a matrix obtained

*This research was supported by grant GACR 201/95/1484.

0024-3795/97/$17.00
PII S0024-3795(96)00004-3

from M by multiplying the columns by suitable positive numbers. The Perron-Frobenius theory of nonnegative matrices yields the fact that this weaker form of diagonal dominance is equivalent to the postulate that all principal minors of M are positive.

This leads in a natural manner to the notion of an M-matrix, introduced by A. Ostrowski in his 1937 paper [13]. The theory of M-matrices enabled Ostrowski to obtain a quantitative refinement of the original result of Hadamard. To formulate it, Ostrowski introduced a Hadamard comparison operator $H(\cdot)$ on the set of all n-by-n matrices as follows: given a complex matrix A, let $H(A)$ be the matrix defined by

$$H(A)_{ii} = |a_{ii}| \quad \text{and} \quad H(A)_{ik} = -|a_{ik}| \quad \text{for } i \neq k.$$

The result of Ostrowski may be formulated as follows:

If $H(A)$ is an M-matrix then $|\det A| \geqslant \det H(A)$.

The lower bound for the modulus of the determinant given by this theorem depends on the moduli of the entries of A only and is, accordingly, the same for the whole class of matrices equimodular with A: Two matrices A_1 and A_2 are called *equimodular* if $H(A_1) = H(A_2)$.

In particular, given A for which $H(A)$ is an M-matrix, all matrices equimodular with A are nonsingular. It is natural to ask whether this implication can be reversed. The following theorem of Camion and Hoffman [2] shows that, up to permutations, the answer is positive:

For a matrix A the following are equivalent:

(i) *every matrix equimodular with A is nonsingular*,
(ii) *there exists a permutation matrix P such that $H(P)$ is an M-matrix.*

It is the purpose of the present note to investigate a Hadamard-type function $A \to H(A)$ for matrices divided into blocks; the mapping $H(\cdot)$ generalizes the classical notion in that it coincides with the classical one for matrices with all blocks one-by-one. It turns out that, using this mapping $H(\cdot)$, it is possible to extend to block matrices both the theorem of Ostrowski and that of Camion and Hoffman.

The methods used to prove the present results are closely related to those used by the authors in their earlier work on M-matrices and on generalized norms. In a series of papers initiated by [6], the authors presented a thorough analysis of the notion of an M-matrix. [In these papers, the class of M-matrices was denoted by K, its closure by K_0; other related classes of matrices were also introduced there, in particular, the class Z (nonpositive off-diagonal entries) and P (positive principal minors). In this manner, $K = Z \cap P$. It seems that the classes Z and P have been universally accepted, but in the case of M-matrices the traditional name remains in use. The advantage of this notation was brevity: An inclusion of the form $M \in K$ requires less space than saying "M is an M-matrix."] In [8], they observed that the logarithm of an M-matrix may be defined in a natural manner and used this fact to analyse the determinantal inequality of Ostrowski, in particular the case of equality. It may seem surprising that the same idea is applicable to block matrices, even with not necessarily square off-diagonal blocks. This is done in Theorem 2 of the present paper.

The possibility of deducing invertibility of a matrix A from the fact that $H(A)$ is an M-matrix has important applications in Gershgorin-type theorems. It is not surprising that attempts to extend such results to block matrices followed immediately; in a natural manner they led to the study of generalized norms on vector spaces. In two independent and almost simultaneous papers of D. G. Feingold, R. S. Varga [4], and the authors [7], the notion of generalized norms and diagonal dominance for block matrices were introduced. In [6], the authors also introduced the notion of a lower bound for a matrix with respect to a generalized norm: If g is a generalized norm on a linear space X (a subadditive positively homogeneous mapping of X into the nonnegative orthant of R_k), a k-by-k matrix H is said to be a lower bound for A if $g(Ax) \geqslant Hg(x)$ for every $x \in X$. If H happens to be invertible and $H^{-1} \geqslant 0$, then clearly A is invertible.

It was to be expected that equivalent properties of matrices used to define a class of matrices like, e.g., M-matrices, will cease being equivalent once they are extended to block matrices. Quite recently, L. Elsner, V. Mehrmann [3] and R. Nabben [11] investigated several classes of block matrices defined by natural extensions of conditions taken from the theory of ordinary M-matrices. Their work gave additional impetus to our investigations.

2. PRELIMINARIES AND NOTATION

If A is an m-by-n matrix, denote by $s_1 \geqslant \cdots \geqslant s_p$ the singular values of A; here, $p = \min(m, n)$ and $s_1^2 \geqslant \cdots \geqslant s_p^2$ are the first p eigenvalues of A^*A or AA^*. Write $s(A)$ for the m-by-n matrix with $s_1 \cdots s_p$ along the diagonal, all other entries being zero. As is well known, there exist unitary matrices U and V such that $A = Us(A)V$.

If B is a block matrix with blocks B_{ik}, we denote by $m(B)$ the block matrix defined by $(m(B))_{ik} = s(B_{ik})$. [If A is considered as a one-block matrix, $m(A)$ coincides with $s(A)$.] Collecting the largest singular values of each block $s_1(A_{ik})$, we obtain a matrix which we denote by $m_*(A)$.

We shall need a notion of equimodularity for block matrices. The following definition is a natural extension of the standard one. Two matrices $A = (A_{ik})$ and $B = (B_{ik})$ partitioned into blocks in the same manner are said to be *equimodular* if $m(A) = m(B)$; in other words, there exist pairs of unitary matrices U_{ik}, V_{ik} of appropriate sizes such that $B_{ik} = U_{ik} A_{ik} V_{ik}$ for each i and k.

If A is an r-by-r block matrix partitioned *symmetrically* into blocks A_{ik} we denote by $H_*(A)$ the r-by-r matrix defined as follows: $(H_*(A))_{ii}$ is the *minimal* singular value of A_{ii}; if $i \neq k$ then $(H_*(A))_{ik} = -s'$, where s' is the *first* (maximal) singular value of A_{ik}.

It is easy to see that if $H_*(A)$ is an M-matrix, then A is nonsingular. Indeed, suppose that $Ax = 0$. If $x = (x_1^T, \dots, x_r^T)^T$ is the corresponding partitioning of x, denote by v the vector $v = (|x_1|, \dots, |x_r|)^T$ where $|x_j| := \|x_j\|_2 = = (x_j^* x_j)^{1/2}$. For each i we have

$$
\begin{aligned}
H_*(A)_{ii}|x_i| &\leqslant |A_{ii}x_i| \\
&= \Big|\sum_{j \neq i} A_{ij}x_j\Big| \\
&\leqslant -\sum_{j \neq i} H_*(A)_{ij}|x_j|.
\end{aligned}
$$

It follows that $H_*(A)v \leqslant 0$; whence, $v = H_*(A)^{-1}(H_*(A)v) \leqslant 0$. Thus $v = 0$ and $x = 0$, which proves that A is nonsingular.

We shall also need another Hadamard-type matrix assigned to a partitioned matrix A. If A is partitioned *symmetrically* into the blocks A_{ik}, we define $H(A)$ as the matrix obtained by replacing the off-diagonal blocks A_{ik} by $-s(A_{ik})$ and defining $H(A)_{ii}$ as the block $s'(A_{ii})$ obtained from $s(A_{ii})$ by reversing the order of the singular values.

Obviously, $H(A)$ may be viewed as a direct sum of matrices H_p consisting of the pth diagonal entries in each block of $H(A)$. The size of H_p thus equals the number of parts in the partitioning that have at least p elements.

3. THE RESULTS

It is the purpose of the present paper to extend to block matrices the two theorems discussed in the introduction.

The extension of the Camion-Hoffman theorem reads as follows:

THEOREM 1. *Let $A = (A_{ik})$ be a square complex matrix partitioned into blocks (not necessarily symmetrically). Then the following are equivalent:*

(i) *every matrix equimodular with A is nonsingular;*

(ii) *there exists a permutation of block rows that takes A into a matrix B with the following properties: the partitioning of B is symmetric, and $H_*(B)$ is an M-matrix.*

For the theorem of Ostrowski the statement of the extension to block matrices remains formally unchanged; of course, the mapping $H(\cdot)$ is now defined for partitioned matrices.

THEOREM 2. *If A is partitioned symmetrically and $H(A)$ is an M-matrix, then*

$$|\det A| \geqslant \det H(A).$$

The proof of Theorem 1 will be given in several steps. We begin by proving the implication (i) → (ii).

Without loss of generality, we may assume that $A = m(A)$. By appropriate permutations of the block rows and of the block columns we can achieve that the sizes $k_1, \dots, k_r$ of the r block rows of the resulting matrix $\tilde{A} = (\tilde{A}_{ik})$ satisfy

$$k_1 \geqslant \cdots \geqslant k_r \geqslant 1$$

and the sizes $l_1, \dots, l_s$ of the s block columns of $\tilde{A}$ satisfy

$$l_1 \geqslant \cdots \geqslant l_s \geqslant 1.$$

Since $\tilde{A}$ is square, say, n-by-n, we have

$$\sum_{i=1}^{r} k_i = \sum_{j=1}^{s} l_j = n.$$

We shall show that $r = s$ and $k_i = l_i$, $i = 1, \dots, r$. For this purpose, we shall reorder the rows and the columns of $\tilde{A}$ as follows: To reorder the rows, we start with the first rows of the blocks, continuing with the second rows of the blocks of size greater than one, etc., and end with the last rows of the blocks having the maximum size. In a similar manner, we reorder the columns of $\tilde{A}$. The result is a block-diagonal matrix, the first diagonal block being $r_1 \times s_1$ with $r_1 = r$, $s_1 = s$. In general, the ith diagonal block is $r_i \times s_i$, where r_i (s_i) is the number of block rows (columns) having at least i rows (columns), $i = 1, \dots, t$, where $t = \min(k_1, l_1)$.

Since this matrix has to be nonsingular, it follows that $r_i = s_i$, $i = 1, \dots, t$, which means that $r = s$ and $k_j = l_j$, $j = 1, \dots, r$.

Let thus

$$
\begin{aligned}
k_1 = k_2 = \cdots = k_{p_1} > k_{p_1+1} = \cdots = k_{p_1+p_2} > \cdots \\
> k_{p_1+p_2+\dots+p_{q-1}+1} = \cdots = k_{p_1+p_2+\dots+p_q},
\end{aligned}
$$

where $p_1 + p_2 + \cdots + p_q = r$, $p_v \geqslant 1$, $v = 1, \dots, q$. For brevity, we shall call the principal block submatrices of $\tilde{A}$ in the first p_1 blocks, in the next p_2 blocks, ..., in the last p_q blocks *intermediate submatrices* of $\tilde{A}$. We shall also say that a square block matrix C with square diagonal blocks is *block-diagonally dominant* if the matrix $H_*(C)$ whose diagonal entries are the smallest singular values of the diagonal blocks and off-diagonal entries the negatives of the greatest singular values of the corresponding off-diagonal blocks is an M-matrix.

We shall now prove several propositions.

PROPOSITION 3.1. *Let*

$$
D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad d_1 > d_2.
$$

Then there exist orthogonal 2-by-2 matrices $U(t)$ and $V(t)$ depending continuously on one real parameter $t \in [0, 1]$ such that

(a) $U(0) = V(0) = I$;

(b) *the* $(1,2)$ *entry of* $U(t)DV(t)$ *is zero for all* $[0, 1]$;

(c) *the* $(1,1)$ *entry of* $U(1)DV(1)$ *is* d_2.

Proof. Choose

$$
c_1 = \cos\frac{\pi}{2}t, \qquad s_1 = \sin\frac{\pi}{2}t,
$$

$$
c_2 = \cos\arctan\left(-\frac{d_2}{d_1}\tan\frac{\pi}{2}t\right), \qquad s_2 = \sin\arctan\left(-\frac{d_2}{d_1}\tan\frac{\pi}{2}t\right),
$$

$$
U(t) = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}.
$$

Properties (a)–(c) are then easily checked. ∎

PROPOSITION 3.2. *Let*

$$d > 0, \qquad C = \begin{pmatrix} d \\ 0 \end{pmatrix}.$$

Then there exists a real unitary 2-by-2 matrix $U(t)$ depending continuously on a real parameter $t \in [-1, 1]$, such that

$$U(t)C = \begin{pmatrix} -td \\ * \end{pmatrix} \tag{1}$$

Proof. Take

$$U(t) = \begin{pmatrix} -t & \sqrt{1-t^2} \\ -\sqrt{1-t^2} & -t \end{pmatrix}. \qquad \blacksquare$$

PROPOSITION 3.3. *Let $A \in M$, $B \geqslant 0$. If $A - tB$ is nonsingular for all $t \in [-1, 1]$, then $A - B$ is an M-matrix.*

Proof. By a well-known property of M-matrices [5, Theorem 5.1] it suffices to show that the spectral radius $r(A^{-1}B)$ is less than one.

Suppose the contrary, i.e., $s = r(A^{-1}B) \geqslant 1$. Since $A^{-1}B \geqslant 0$, the Perron-Frobenius theorem yields that s is an eigenvalue of $A^{-1}B$: thus $\det(sI - A^{-1}B) = 0$. Therefore, $\det[A - (1/s)B] = 0$, which contradicts our assumption. $\blacksquare$

The last proposition describes the intermediate submatrices.

PROPOSITION 3.4. *In each intermediate submatrix of $\tilde{A}$ there exists a permutation of block rows such that the resulting matrix is block-diagonally dominant.*

Proof. For $w \in \{1, \dots, q\}$, denote by m_w the sum $m_w = \sum_{i=1}^{w} p_i$. Let $v \in \{1, \dots, q\}$. We shall show first that the matrix formed by the (1,1) entries of all the blocks in the intermediate submatrix C_v of $\tilde{A}$ with indices in $\mathscr{I}_v = \{m_{v-1} + 1, \dots, m_v\}$ satisfies the assumption of the Camion-Hoffman theorem.

Observe that all blocks $\tilde{A}_{\alpha\beta}$ with $1 \leqslant \beta \leqslant m_{v-1}$ and $m_{v-1} < \alpha \leqslant r$ have more columns than rows. Therefore, by multiplication by an appropriate permutation matrix $V_{\alpha\beta}$ one can arrange that the first column in each such block $\tilde{A}_{\alpha\beta}V_{\alpha\beta}$ will be zero. In each of the first columns of the first m_{v-1} block columns, there are nonzero entries in the first rows of the first m_{v-1} block rows only. Consequently, the determinant in these rows and columns can be factored out from the determinant of the whole matrix. After deleting these rows and columns, the remaining submatrix then has zero columns in the first column of each of the blocks $\tilde{A}_{\alpha\beta}$ for $1 \leqslant \alpha \leqslant m_{v-1}$ and $\beta \in \mathscr{I}_v$. If we multiply the blocks $\tilde{A}_{\alpha\beta}$, as above, by appropriate unitary matrices $V_{\alpha\beta}$ for $\alpha > m_v$ and $\beta \in \mathscr{I}_v$, the first columns in these new $\tilde{A}_{\alpha\beta}$'s will be zero. It follows that the determinant formed by the $(1,1)$ entries in the blocks $\tilde{A}_{\alpha\beta}$ for $\alpha, \beta \in \mathscr{I}_v$ can be factored out from the determinant of one of the equimodular matrices and thus, by Theorem 1(i), is different from zero. The same is true even if we multiply (independently) each entry of this last determinant by a complex unit. By the Camion-Hoffman theorem there exists a permutation of the block rows with indices in $\mathscr{I}_v$ such that for the resulting matrix, the comparison matrix of the submatrix of the $(1,1)$ entries is an M-matrix. If $k_j = 1$ for $j \in \mathscr{I}_v$, we are finished. If $k_j > 1$ for $j \in \mathscr{I}_v$, one can arrange, staying in the set of equimodular matrices, that in each block $\tilde{A}_{pq}$, $p, q \in \mathscr{I}_v$, the second entry in the main diagonal of $\tilde{A}_{pq}$ is the smallest singular value. Using Proposition 3.1 for the first two rows and columns, we can proceed continuously to the case that the $(1,1)$ entries of the diagonal blocks are the smallest singular values and those of the off-diagonal blocks the negatives of the greatest singular values, and the matrix will still be an M-matrix. The proof of Proposition 3.4 is complete. ■

Let us return to the proof of the implication (i) → (ii). By Proposition 3.4, we can arrange that all intermediate submatrices are already block-diagonally dominant. We shall show that then the whole resulting matrix $\hat{A}$ is block-diagonally dominant.

Let us consider the $(1,1)$ entries of all the blocks of the matrix $\hat{A}$. We already know that all comparison matrices corresponding to the intermediate submatrices are M-matrices. These matrices together form thus an r-by-r M-matrix A_0. None of the blocks not belonging to intermediate submatrices is square. The blocks $\tilde{A}_{pq}$ in the upper triangular part (for $p < q$) have more rows than columns, their $(1,1)$ entry σ_{pq} is the corresponding greatest singular value, and their last row is a zero row. By Proposition 3.2, we can arrange in each of these blocks, by a multiplication from the left by a unitary matrix depending continuously on a real parameter from the interval $[-1,1]$, that their $(1,1)$ entries have the form $-t\sigma_{pq}$ for $t \in [-1,1]$. The same can be done in the blocks in the lower triangular part.

Denote by B the r-by-r matrix which has its off-diagonal entries b_{pq} for p, q not belonging to the same interval $\mathscr{I}_v$ equal to σ_{pq} and all the remaining entries zero. All matrices of the form $A_0 - tB$ are by (i) nonsingular, since they form one block of the reducible transformed matrix equimodular to $\tilde{A}$. By Proposition 3.3, $A_0 - B$ is an M-matrix. Since it is equal to $H_*(\hat{A})$, the proof of the implication (i) → (ii) is complete.

In the proof of the implication (ii) → (i) we shall use the theory of generalized norms. Suppose that $A = (A_{ik})$ is already appropriately permuted, has r block rows, and is n-by-n. The partitioning of A corresponds thus to the partitioning of the linear space X of column n-vectors into a direct sum of r subspaces $X_1, \dots, X_r$, with X_i consisting of column vectors with nonzero coordinates in the ith block only. Let g_i be the l_2-norm on X_i, $i = 1, \dots, r$, and let

$$g_{ij}(A_{ij}) = \sup_{x_j \neq 0,\, x_j \in X_j} \frac{g_i(A_{ij}x_j)}{g_j(x_j)},$$

$$\hat{g}_{ii}(A_{ii}) = \inf_{x_i \neq 0,\, x_i \in X_i} \frac{g_i(A_{ii}x_i)}{g_i(x_i)}.$$

Since $g_{ij}(A_{ij})$ is the greatest singular value of A_{ij} and $\hat{g}_{ii}(A_{ii})$ the smallest singular value (which can be zero) of the square matrix A_{ii}, the matrix $\hat{G}$ with diagonal entries $\hat{g}_{ii}(A_{ii})$ and off-diagonal entries $-g_{ij}(A_{ij})$ coincides with $H_*(A)$ and by (ii) is an M-matrix.

To show that A is nonsingular, let $x \in X$ be such that

$$Ax = 0, \qquad x = \left(x_1^T, \dots, x_r^T\right)^T, \quad x_i \in X_i, \ i = 1, \dots, r.$$

The generalized norm [7] $g(\cdot)$ defined as

$$g(x) = \left(g_1(x_1), \dots, g_r(x_r)\right)^T$$

satisfies

$$g(Ax) = 0.$$

Since

$$Ax = \begin{pmatrix} \Sigma_k A_{1k}x_k \\ \vdots \\ \Sigma_k A_{rk}x_k \end{pmatrix}$$

and

$$g_i\left(\sum_k A_{ik}x_k\right) \geqslant \hat{g}_{ii}(A_{ii})g_i(x_i) - \sum_{k\neq i} g_{ik}(A_{ik})g_k(x_k),$$

we obtain

$$g(Ax) \geqslant \hat{G}g(x),$$

i.e.

$$0 \geqslant \hat{G}g(x).$$

Since $\hat{G}^{-1}$ is an M-matrix, $\hat{G}^{-1}$ is nonnegative, so that

$$0 \geqslant g(x),$$

whence $g(x) = 0$, and $x = 0$.

Thus A is nonsingular, and the same holds for all matrices $(U_{ik}A_{ik}V_{ik})$, since $\hat{G}$ is the same. Therefore, (i) holds.

This completes the proof of Theorem 1. ■

We now proceed to prove Theorem 2; one of the tools used is a generalization of an inequality of von Neumann [12].

PROPOSITION 3.5. *If $A_1, \dots, A_s$ are complex matrices for which the product $A_1 \cdots A_s$ exists and is a square matrix, then*

$$\begin{aligned}\operatorname{Re}\operatorname{tr}(A_1 \cdots A_s) &\leqslant |\operatorname{tr}(A_1 \cdots A_s)| \leqslant \operatorname{tr} m(A_1 \cdots A_s)\\ &\leqslant \operatorname{tr}[m(A_1)\cdots m(A_s)]. \end{aligned} \tag{2}$$

Proof. Let M be the maximum of the dimensions of all the matrices $A_1, \dots, A_s$. Denote, for a moment, by $\tilde{A}_k$ the $M \times M$ matrix

$$\tilde{A}_k = \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad k = 1, \dots, s.$$

Since $\operatorname{tr}(\tilde{A}_1 \cdots \tilde{A}_s) = \operatorname{tr}(A_1 \cdots A_s)$, etc, it suffices to prove (2) in the case that all matrices $A_1, \dots, A_s$ are square. For $s = 2$, (2) is von Neumann's inequality [12, Theorem 1; 10, Theorem 2]. The case $s = 1$ follows from this for $A_2 = I$. Let us prove now that for $n \times n$ matrices A, B, C

$$\operatorname{tr}[m(AB)m(C)] \leqslant \operatorname{tr}[m(A)m(B)m(C)]. \tag{3}$$

It is well known (see e.g. [9, Chapter 9, H2]) that the ordered singular values $\sigma_1 \geqslant \cdots \geqslant \sigma_n$ satisfy

$$\sum_{i=1}^{k} \sigma_i(AB) \leqslant \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B), \qquad k = 1, \dots, n.$$

If $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ are the singular values of C, multiply these inequalities by $\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \dots, \gamma_{n-1} - \gamma_n, \gamma_n$ and add them. We obtain

$$\sum_{i=1}^{n} \gamma_i \sigma_i(AB) \leqslant \sum_{i=1}^{n} \gamma_i \sigma_i(A)\sigma_i(B),$$

which is (3). It follows that for $s > 2$

$$\begin{aligned}
\operatorname{Re}\operatorname{tr}(A_1 \cdots A_s) &\leqslant \operatorname{tr} m(A_1 \cdots A_{s-1}A_s) \\
&\leqslant \operatorname{tr}[m(A_1 \cdots A_{s-1})m(A_s)] \\
&\leqslant \operatorname{tr}[(m(A_1 \cdots A_{s-2})m(A_{s-1})m(A_s))].
\end{aligned}$$

An easy induction completes then the proof of (2). ■

In Proposition 3.5, each matrix A_k was considered as a single block. In the following proposition we shall show that a similar result holds for matrices with more blocks [which influences the matrices $m(\cdot)$].

PROPOSITION 3.6. *Let $B_1, \dots, B_t$ be square block matrices symmetrically partitioned in the same way. Then*

$$\begin{aligned}
\operatorname{Re}\operatorname{tr}(B_1 \cdots B_t) \leqslant |\operatorname{tr}(B_1 \cdots B_t)| &\leqslant \operatorname{tr} m(B_1 \cdots B_t) \\
&\leqslant \operatorname{tr}[m(B_1) \cdots m(B_t)],
\end{aligned} \tag{4}$$

where $m(B_k)$ means the block matrix defined above.

Proof. In the product $B_1 \cdots B_t$, each diagonal block [only such occur in the inequalities (4)] is a sum of products of the blocks in the factors. Applying (2) to separate summands, (4) follows. ∎

PROPOSITION 3.7. *Let B be a symmetrically partitioned square matrix. Then the spectral radii of B and of $m(B)$ are related by the estimate*

$$r(B) \leqslant r(m(B)).$$

Proof. For any square matrix C,

$$r(C) = \limsup |\operatorname{tr} C^k|^{1/k}.$$

Thus by (2)

$$\begin{aligned} r(B) &= \limsup |\operatorname{tr} B^k|^{1/k} \\ &\leqslant \limsup \left\{\operatorname{tr}\left[m(B)\right]^k\right\}^{1/k} \\ &= r(m(B)). \end{aligned}$$ ∎

PROPOSITION 3.8. *Let B be a symmetrically partitioned block square matrix. If $r(m(B)) < 1$ then*

$$|\det(I - B)| \geqslant \det[I - m(B)].$$

Proof. By Proposition 3.7, $r(B) < 1$. It follows that the series

$$\sum_{s=1}^{\infty} s^{-1} B^s$$

converges and its sum M satisfies

$$I - B = \exp(-M).$$

Therefore,

$$\begin{aligned} |\det(I - B)| &= |\exp \operatorname{tr}(-M)| \\ &= \exp(-\operatorname{Re} \operatorname{tr} M) \\ &= \exp\left(-\sum s^{-1} \operatorname{Re} \operatorname{tr} B^s\right). \end{aligned}$$

Since

$$\operatorname{Re}\operatorname{tr} B^s \leqslant \operatorname{tr} m(B)^s$$

by (2), and since $r(m(B)) < 1$, the series $\sum_{s=1}^{\infty} s^{-1} \operatorname{tr}[m(B)]^s$ converges and

$$\begin{aligned}\exp\left(-\sum s^{-1} \operatorname{Re}\operatorname{tr} B^s\right) &\geqslant \exp\left(-\sum s^{-1} \operatorname{tr} m(B)^s\right) \\ &= \det[I - m(B)].\end{aligned}$$

■

Proof of Theorem 2. There exist two block-diagonal matrices U and V consisting of unitary blocks such that the diagonal blocks of $A' = UAV$ will be diagonal matrices. Let k be a real number not smaller than the maximum of all the diagonal entries of A', and define B by the relation $A' = k(I - B)$. Observe that $|\det A| = |\det A'|$ and that $H(A) = H(A') = k[I - m(B)]$. By Proposition 3.7 we have $r(B) \leqslant r(m(B))$; since $H(A)$ is an M-matrix, $r(m(B)) < 1$. Thus $r(B) < 1$, and Proposition 3.8 yields

$$\begin{aligned}|\det A'| &= k^n |\det(I - B)| \\ &\geqslant k^n \det[I - m(B)] \\ &= \det H(A').\end{aligned}$$

The proof is complete. ■

REFERENCES

1 T. Ando, Inequalities for M-matrices, *Linear and Multilinear Algebra* 8:291–316 (1980).

2 P. Camion and A. J. Hoffman, On the nonsingularity of complex matrices, *Pacific J. Math.* 17:211–214 (1966).

3 L. Elsner and V. Mehrmann, Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations, *Numer. Math.* 59:541–559 (1991).

4 D. G. Feingold and R. Varga, Block diagonally dominant matrices and generalizations of the Gershgorin circle theorem, *Pacific J. Math.* 12:1241–1250 (1963).

5 M. Fiedler, *Special Matrices and Their Applications in Numerical Mathematics*, Martinus Nijhoff, Dordrecht, 1986.

6 M. Fiedler and V. Pták, On matrices with non-positive off-diagonal elements and positive principal minors, *Czechoslovak Math. J.* 87:382–400 (1962).
7 M. Fiedler and V. Pták, Generalized norms of matrices and the location of the spectrum. *Czechoslovak Math. J.* 87:558–571 (1962).
8 M. Fiedler and V. Pták, Cyclic products and an inequality for determinants, *Czechoslovak Math. J.* 94:428–451 (1969).
9 A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic, New York, 1979.
10 L. Mirsky, On the trace of matrix products, *Math. Nachr.* 20:171–174 (1959).
11 R. Nabben, On a class of matrices which arise in the numerical solution of Euler equations, *Numer. Math.* 63:411–431 (1992).
12 J. von Neumann, Some matrix inequalities and metrization of matrix space, *Tomsk Univ. Rev.* 1:286–299 (1937).
13 A. M. Ostrowski, Über Determinanten mit überwiegender Diagonale, *Comment. Math. Helv.* 10:69–96 (1937).
14 O. Taussky, A recurring theorem on determinants, *Amer. Math. Monthly* 56:672–675 (1949).
15 R. S. Varga, On recurring theorems on diagonal dominance. *Linear Algebra Appl.* 13:1–9 (1976).

Received 16 June 1993; final manuscript accepted 28 November 1995